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Multifractal Spectra of Fragmentation Processes

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Let $(S(t), t \geqslant 0)$ be a homogeneous fragmentation of $]0, 1[$ with no loss of mass. For $x \in]0, 1[$, we say that the fragmentation speed of x is v if and only if, as time passes, the size of the fragment that contains x decays exponentially with rate v. We show that there is $v_{\text{typ}} > 0$ such that almost every point $x \in]0, 1[$ has speed v_{typ}. Nonetheless, for v in a certain range, the random set $\mathscr{G}_v$ of points of speed v, is dense in $]0, 1[$, and we compute explicitly the spectrum $v \to \text{Dim}(\mathscr{G}_v)$ where Dim is the Hausdorff dimension.

KEY WORDS: Fragmentation; Galton–Watson trees; multifractal spectra.

1. INTRODUCTION

Fragmentation underlies a number of physical, chemical and geological processes, such as polymer degradation,[1, 2] liquid droplet breakup,[3] crushing or grinding of rocks,[4] atomic collisions and nuclear multifragmentation,[5] or energy cascade in turbulence to name just a few. One can also report to the proceedings[6] for some applications in physics and to refs. 7, 8, and 9 for computer science. The fragmented quantity in such processes are diverse: mass, momentum, energy, or surface. But a global characteristic feature is that each new fragment continues splitting independently. Usually the simplifying assumption that each fragment can be described by a single state variable (e.g., mass) is also made.

Informally, the purpose of this work is to investigate the set of locations having an abnormally fast (or slow) fragmentation speed in a so-called homogenous fragmentation of a one dimensional object.

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1.1. An Example

Let us first introduce *via* a simple example the ideas and notions on which this paper focuses. A fragmentation model describes an object endowed with a unit mass measure that falls apart as time runs. We consider the following example which is a continuous version of a model due to Kolmogoroff in what seems to be one of the first probabilistic work on random fragmentations (see ref. 10). Define $S(t)$ to be a Markov process with values in the space $\mathcal{O}$ of the open subsets of $]0, 1[$, which starts from $S(0) =]0, 1[$ and evolves as follows. Each segment of length m is cut in two with rate 1 (i.e., after an exponential time with mean 1). It then gives rise to two new segments (which are thus included in their "father") of respective length Vm and $(1-V)\,m$, where V is a random variable with values in $]0, 1[$ independent of the past. The new segments follow the same dynamics independently. The process $S(t)$ is called a homogeneous *interval fragmentation*.

For a given point $x \in]0, 1[$ and a realization of an interval fragmentation we will say that x has *fragmentation speed* or *rate of decrease* v if

$$-\log(|I_x(t)|)/t \to v$$

where $|I_x(t)|$ is the size of the fragment that contains x at time t. Pick $u \in]0, 1[$ at random from the uniform distribution, then it is plain from our construction that $|I_u(t)|$ is a pure jump process, and that the waiting times between each jump are exponential with parameter 1. Clearly $\xi(t) = -\log(|I_u(t)|)$ is a compound Poisson process whose increment is given by the random variable $-\log(Z)$ where conditionally on V, Z is V with probability V and is $1-V$ with probability $1-V$. Thus, if we define $v_{\text{typ}} := \mathbf{E}(-\log(Z))$, then almost surely

$$-\frac{1}{t}\log|I_u(t)| \to v_{\text{typ}} \qquad \text{as} \quad t \to \infty.$$

Hence almost every points in $]0, 1[$, in the sense of the Lebesgue measure, has rate of decrease v_{typ}, which is thus the *typical* rate of decrease.

On the other hand, we can define the process $J(t)$ of "the largest fragment followed" inductively by following the largest fragment at each dislocation. Again $-\log(J(t))/t$ is a compound Poisson process and the SLLN entails the almost sure convergence of $-\log(J(t))/t \to v'$ where clearly $v' < v_{\text{typ}}$. So there must be some points whose fragmentation speed is less than v_{typ}. The same technique also allows us to find some fast points of fragmentation by selecting the smallest fragment at each dislocation.

If we give ourselves an interval $]a, b[$ in $]0, 1[$, then almost surely there exists a time t such that one of the interval components of $S(t)$ is included in $]a, b[$. As new segments evolve independently and with the same dynamics as the original fragment, the same analysis applies and $]a, b[$ contains almost surely a point with fragmentation speed v'. This shows that if there exists almost surely a point whose fragmentation speed is $v > 0$, then there is a dense subset of $]0, 1[$ of points having the same property.

Although the example above is good for intuition, it has two limits. First one can suppose that when a segment splits, it might give birth to any number of sub-segments, possibly infinite, and not just two. Second, in this example, the splitting times are "discrete," the first time of splitting is almost surely strictly positive. In the sequel we consider more generally the case where fragmentation may occur continuously.

The aim of this paper is to show that, for a large class of homogeneous fragmentations, there are some points with a different fragmentation speed than the typical one and to study the sets of such points. This problem resembles by some aspects the study of the fast points of the Brownian motion of Orey and Taylor[11] (see also Davis,[12] Kahane,[13] and Perkins[14] for more insight on fast and slow points of the Brownian motion) or the recent works of Dembo et al. on thin and thick points of planar and d-dimensional Brownian motion (see refs. 15–17). There are also some natural and obvious connections with the theory of branching processes that stem from the fact that there is a genealogic structure in the interval fragmentations. Some of the techniques we use are close to those used by Shieh and Taylor,[18] Shieh and Mörters,[19] and Liu[20] for studying the multifractal spectra of the branching measure. This connection was already used in the analysis of self-similar recursive fractal, see Hambly and Jones[21] and the references therein. Although we do not make use of it here, there seems to be a way of doing some of the proofs using ideas of percolation on a tree (see refs. 22 and 23 or the book in preparation[24]).

We now introduce informally some notions related to fragmentations (definitions being given in Section 2) and state our main results.

1.2. Main Results

A homogeneous interval fragmentation $S(t)$ can be heuristically described as a nested family of open subsets of $]0, 1[$ (i.e., $S(t) \subseteq S(s)$ whenever $t \geqslant s$) such that when a new fragment appears, it starts a new independent fragmentation which, up to the scale factor, has the same law as the initial one.

Define

$$\mathscr{S}^{\downarrow} := \left\{ x = (x_1, x_2, \ldots),\ x_1 \geqslant x_2 \geqslant \cdots \geqslant 0,\ \sum_i x_i \leqslant 1 \right\}.$$

Denote by Λ the map $\mathscr{O} \to \mathscr{S}^{\downarrow}$ that associates to an open subset of $]0, 1[$ the ordered vector of the lengths of its interval decomposition. Then if $S(t)$ is an interval fragmentation, we denote by $X(t) = \Lambda(S(t))$ the associated *ranked fragmentation*.[2] The configuration space for ranked fragmentations is $\mathscr{S}^{\downarrow}$.

We suppose that at all times the Lebesgue measure of $S(t)$ is 1 (i.e., there is no loss of mass), thus the associated ranked fragmentation $X(t) = \Lambda(S(t))$ takes its values in the smaller space

$$\mathscr{S}_1^{\downarrow} := \left\{ x = (x_1, x_2, \ldots),\ x_1 \geqslant x_2 \geqslant \cdots \geqslant 0,\ \sum_i x_i = 1 \right\}.$$

Bertoin has shown in ref. 25 that the law of the process $X(t)$ is completely characterized by the so-called splitting measure $\nu(.)$ which is a measure on $\mathscr{S}_1^{\downarrow}$ such that

$$\int_{\mathscr{S}_1^{\downarrow}} (1 - x_1)\, \nu(dx) < \infty. \tag{1}$$

Roughly, the splitting measure describes the "rates" at which fragments split. Heuristically, if $A \subset \mathscr{S}^{\downarrow}$, then, for any $r \in]0, 1[$, $\nu(A)$ is the rate at which a fragment of size r splits into smaller masses $(x_1, x_2, \ldots)$ such that $(x_1/r, x_2/r, \ldots) \in A$.

Some information about ν is caught by the function Φ:

$$\Phi(q) := \int_{\mathscr{S}_1^{\downarrow}} \left(1 - \sum_{i=1}^{\infty} x_i^{q+1} \right) \nu(dx), \qquad q > \underline{p}, \tag{2}$$

where

$$\underline{p} := \inf \left\{ p \in \mathbb{R} : \int_{\mathscr{S}_1^{\downarrow}} \sum_{i=2}^{\infty} x_i^{p+1} \nu(dx) < \infty \right\}.$$

From now on, we always suppose $\underline{p} < 0$.

[2] Conversely, it is known that given $X(t)$ there exists $S(t)$ such that $X(t) \overset{L}{=} \Lambda(S(t))$ in the sense of identity of the finite dimensional marginals.

As in ref. 26, define two constants

$$v_{\text{typ}} := \Phi'(0+) = -\int_{\mathscr{S}_1^\downarrow} \left(\sum_{i=1}^{\infty} x_i \log(x_i) \right) \nu(dx)$$

which, as we shall see, is the typical rate of decrease, i.e., it is a.s. the fragmentation speed of a point picked at random in $]0, 1[$ from the uniform distribution, and $v_{\min}$ as follows. Let $\bar{p} > 0$ be the unique solution of the equation

$$(p+1)\,\Phi'(p) = \Phi(p), \qquad p > \underline{p}.$$

The function $p \to \Phi(p)/(p+1)$ reaches its unique maximum $v_{\min}$ on $]\underline{p}, \infty[$ at $\bar{p}$ and

$$v_{\min} := \Phi'(\bar{p}) = \Phi(\bar{p})/(\bar{p}+1).$$

It is shown in ref. 26 that $v_{\min}$ is the rate of exponential decrease of the largest fragment, i.e., almost surely

$$\lim_{t \to \infty} -t^{-1} \log X_1(t) = v_{\min}.$$

So clearly $v_{\min} \leqslant v_{\text{typ}}$. However this does not mean that there exists some point with rate $v_{\min}$: for that we would need the existence of a point which is often enough in the largest fragment, and such a point might well not exist. Define also

$$v_{\max} := \Phi'(\underline{p}+) \in [0, \infty].$$

Let $\mathscr{G}_v$ be the set of points with fragmentation speed v

$$\mathscr{G}_v := \{x \in]0, 1[: \lim_{t \to \infty} -t^{-1} \log(|I_x(t)|) = v\}.$$

Also define $\overline{\mathscr{G}}_v$ and $\underline{\mathscr{G}}_v$ as follows:

$$\overline{\mathscr{G}}_v := \{x \in]0, 1[: \limsup_{t \to \infty} -t^{-1} \log(|I_x(t)|) \leqslant v\},$$

$$\underline{\mathscr{G}}_v := \{x \in]0, 1[: \liminf_{t \to \infty} -t^{-1} \log(|I_x(t)|) \geqslant v\}.$$

Thus a point in $\underline{\mathscr{G}}_v$ (resp. in $\overline{\mathscr{G}}_v$) will be, for t large enough, in a small (resp. large) fragment compared to e^{-vt}.

Let Υ_v be the reciprocal of v by Φ', i.e., $\Phi'(\Upsilon_v) = v$. Define

$$C(v) := (\Upsilon_v + 1)\, v - \Phi(\Upsilon_v) \tag{3}$$

for $v \in]v_{\min}, v_{\max}[$ and $C(v) = -\infty$ elsewhere. As $\tilde{C}(v) = C(v) - v$ is thus the Legendre transform $\mathscr{L}$ of Φ, one has

$$\mathscr{L}\tilde{C}(.) = \mathscr{L}\mathscr{L}\Phi(.) = \Phi(.).$$

Thus Φ and C determine each other uniquely.

A fragmentation such that each splitting produces two fragments (i.e., ν only charges the subspace $\{x \in \mathscr{S}_1^{\downarrow} : x_3 = 0\}$) is called a *binary fragmentation*. In the case of a binary fragmentation, it is not hard to see that Φ determines ν, thus Φ characterizes the law of the fragmentation.

Define the *fragmentation spectra* to be, for $v_{\min} < v < v_{\max}$, the function that associates the Hausdorff dimension (denoted by Dim(.) in the sequel) of the sets $\mathscr{G}_v$, $\overline{\mathscr{G}}_v$, and $\underline{\mathscr{G}}_v$ to v (for a definition and main properties of the Hausdorff measure and dimension see, for instance, ref. 27).

The following theorem, which gives the fragmentation spectra explicitly in terms of the function $v \to C(v)$ thus entails that the law of a binary fragmentation is characterized by its spectra.

Theorem 1. For each $v \in]v_{\min}, v_{\max}[$, almost surely

$$\mathrm{Dim}(\mathscr{G}_v) = C(v)/v = 1 + \Upsilon_v - \Phi(\Upsilon_v)/v, \tag{4}$$

$$\mathrm{Dim}(\overline{\mathscr{G}}_v) = C(v)/v \quad \text{if} \quad v \leqslant v_{\mathrm{typ}} \quad \text{and} \quad = 1 \quad \text{if} \quad v \geqslant v_{\mathrm{typ}}, \tag{5}$$

$$\mathrm{Dim}(\underline{\mathscr{G}}_v) = C(v)/v \quad \text{if} \quad v \geqslant v_{\mathrm{typ}} \quad \text{and} \quad = 1 \quad \text{if} \quad v \leqslant v_{\mathrm{typ}}. \tag{6}$$

One can easily verify that $C(v_{\mathrm{typ}})/v_{\mathrm{typ}} = 1$, $C(v_{\min})/v_{\min} = 0$, and that $C(v)/v$ is continuous and decreases as v get farther of v_{typ}.

However, we do not necessarily have $C(v)/v \to 0$ when $v \to v_{\max}$.

More precisely as

$$\lim_{v \to v_{\max}} C(v)/v = \lim_{p \searrow \underline{p}} 1 + p - \Phi(p)/\Phi'(p) \geqslant 1 + \underline{p},$$

as soon as $\underline{p} > -1$, for each $v > v_{\mathrm{typ}}$ one has that

$$\mathrm{Dim}(\underline{\mathscr{G}}_v) \geqslant 1 + \underline{p} > 0.$$

The natural question in this setting is "are there some points with a superexponential fragmentation behavior?" or more precisely, can we define a set of such points with a non-trivial dimension?

Consider the set

$$\mathscr{H} = \left\{ x \in]0, 1[: \limsup_{t \to \infty} -\frac{1}{t} \log |I_x(t)| = +\infty \right\}.$$

Then it can be shown in the case $\Phi'(\underline{p}+) = \infty$ that

$$\mathrm{Dim}(\mathscr{H}) = 1 + \underline{p}.$$

The upper bound is established as in the case of $\mathscr{G}_v$ (see Section 3) and the lower bound can be obtained roughly through the same techniques employed in Section 4.

The rest of this paper is organized as follows. The next section introduces notations, notions, and definitions. Upper bounds are given in Section 3. Section 4, which represents the most important part of this work, gives a lower bound for the Hausdorff dimension of $\mathscr{G}_v$ using a Galton–Watson tree that reflects the genealogical structure of the interval fragmentation.

2. PRELIMINARIES

We now recall some facts mostly lifted from refs. 25, 26, 28, and 29 on homogeneous fragmentations and their asymptotic behavior.

A homogeneous interval fragmentation is a Markov process with values in $\mathscr{O}$ which enjoys two key properties: fragmentation and homogeneity. The fragmentation property states that when a new fragment (a new segment in the example) is born, it starts a new independent fragmentation of its own. This can be seen as a version of the branching property. The homogeneity property states that this new fragmentation has, up to a scaling factor, the same law as the initial one.

Specifically, if $\mathbb{P}$ stands for the law of the interval fragmentation $S(t)$ started from $]0, 1[$, then for $s, t \geqslant 0$ conditionally on

$$S(t) = \bigcup_{i \in \mathbb{N}} J_i(t)$$

(where $J_i(t)$ is the interval decomposition of the open $S(t)$, i.e., for each i, $J_i(t)$ is an open subinterval of $S(t)$, the J_i's are disjoints and $\bigcup_i J_i(t) = S(t)$) $S(t+s)$ has same law as $S^{(1)}(s) \cup S^{(2)}(s) \cup \cdots$ where for each i, $S^{(i)}(s)$ is a subset of $J_i(t)$ and has same distribution as the image of $S(s)$ by the homothetic map $]0, 1[\to J_i(t)$.

Similarly, a homogeneous ranked fragmentation is a Markov process with values in $\mathscr{S}_1^{\downarrow}$ such that if $\mathbb{P}$ stands for the law of the ranked fragmentation X started from $(1, 0, 0,...)$, then for $s, t \geqslant 0$ conditionally on $X(t) = (x_1, x_2,...)$, $X(t+s)$ has same law as the variable obtained by reordering the

elements of the random sequences $X^{(1)}(s), X^{(2)}(s),...,$ where for each i, $X^{(i)}(s) \in \mathscr{S}^{\downarrow}$ has same distribution as $X(s)$ under $\mathbb{P}_{x_i}$ where $\mathbb{P}_r$ is the image of $\mathbb{P}$ by the map

$$(x_1, x_2,...) \to (rx_1, rx_2,...).$$

The homogeneous fragmentations we shall consider in this work are those for which there is no loss of mass, i.e., such that almost surely, for all $t > 0$

$$\sum_i X_i(t) = 1.$$

This is why the configuration space is $\mathscr{S}_1^{\downarrow}$ and not, as usual, the more general space $\mathscr{S}^{\downarrow}$. As we have said, the law of such a ranked fragmentation is completely characterized by a so called splitting measure, ν, which is a measure on $\mathscr{S}_1^{\downarrow}$ that verifies the integral condition (1).

The interpretation of the function Φ given by (2) is the following. Suppose that at the initial time, a random point u with uniform distribution is tagged on $]0, 1[$: as in the example $|I_u(t)|$ is a size biased pick from $X(t) = (X_1(t), X_2(t),...) = \Lambda(S(t))$, i.e.,

$$|I_u(t)| \overset{L}{=} X_K(t)$$

where K is a random variable with values in $\mathbb{N}$ such that

$$P(K = k \mid X(t)) = X_k(t), \qquad k = 1,... .$$

Then the process

$$\xi(t) = -\log(|I_u(t)|) \tag{7}$$

is a subordinator (an increasing Lévy process) and we have

$$\mathbf{E}(|I_u(t)|^q) = \mathbf{E}(e^{-q\xi(t)}) = e^{-t\Phi(q)}, \qquad t \geqslant 0,$$

where Φ is given by (2) (see ref. 25 for the proof and discussion). This has direct consequences such as $\Phi :]\underline{p}, \infty[\to]-\infty, \infty[$ being the Laplace exponent of a subordinator, it is a concave increasing analytic function. Furthermore $\Phi(0+) = 0$ (this comes from the mass conservation). Remark that

$$v_{\text{typ}} := \mathbf{E}(\xi(1)).$$

Then by the L.L.N., if $v_{\text{typ}} < \infty$ (which holds whenever $\underline{p} < 0$), for Lebesgue almost every point $x \in]0, 1[$

$$\lim_{t \to \infty} -t^{-1} \log(|I_x(t)|) = v_{\text{typ}} \quad \text{a.s.}$$

which proves that $|\mathscr{G}_{v_{\text{typ}}}|$, the Lebesgue measure of $\mathscr{G}_{v_{\text{typ}}}$, is 1.

The starting point of this work is an estimate obtained by Bertoin in ref. 26 concerning the number of abnormally "large" or "small" fragments at time t. More precisely, consider a homogeneous ranked fragmentation $(X_t)_{t \geqslant 0}$, then, for $v \in]v_{\min}, v_{\max}[$ one has with probability one

$$\lim_{\epsilon \to 0} \lim_{t \to \infty} t^{-1} \log(\text{Card}\{i \in \mathbb{N} : e^{-(v+\varepsilon)t} \leqslant X_i(t) \leqslant e^{-(v-\varepsilon)t}\}) = C(v), \qquad (8)$$

where $C(v)$ is the function defined by (3).

We proceed in two steps to prove Theorem 1: we will first give upper bounds and then a lower bound for $\mathscr{G}_v$ and use inclusions to conclude.

3. UPPER BOUND

We prove the upper bound for the dimension of $\underline{\mathscr{G}}_v$ and $\overline{\mathscr{G}}_v$ and the conclusion follows for $\mathscr{G}_v$ by the inclusion

$$\mathscr{G}_v \subset \underline{\mathscr{G}}_v \cap \overline{\mathscr{G}}_v.$$

Let $(S(t))_{t \geqslant 0}$ be an interval fragmentation. We denote by

$$X(t) = (X_1(t), X_2(t), \ldots) = \Lambda(S(t))$$

the associated ranked fragmentation. We also use the notation

$$S(t) = \bigcup_{i \in \mathbb{N}} J_i(t)$$

where $(J_1(t), J_2(t), \ldots)$ is an interval decomposition of $S(t)$ and the labelling is size-wise, i.e., for each i, $X_i(t) = |J_i(t)|$.

3.1. Upper Bound for (6)

In this section we consider the case $v > v_{\text{typ}}$. Define the collection of indices

$$\Theta_v(t) = \{i \in \mathbb{N} : X_i(t) \leqslant e^{-vt}\}$$

and note that $\forall N \in \mathbb{N}$, for all $w \in]v_{\text{typ}}, v[$ the set $\bigcup_{n \geqslant N} \bigcup_{i \in \Theta_w(n)} J_i(n)$ is a cover of $\underline{\mathscr{G}}_v$ (actually it is a cover for the larger set

$$\left\{ x \in]0, 1[: \limsup_{t \to \infty} -\frac{1}{t} \log |I_x(t)| > v \right\}$$

and hence the upper-bound for the dimension of $\mathscr{H}$ is also proven here). Thus we want to show that for $\alpha > C(v)/v$ as closed to $C(v)/v$ as wished, for w close enough to v

$$\sum_n \sum_{i \in \Theta_w(n)} X_i^\alpha(n) < \infty$$

Fix $\epsilon > 0$ and take $\alpha = C(v)/v + \epsilon$.

Clearly, for any $\beta \in [0, \alpha[$

$$\sum_{i \in \Theta_w(n)} X_i^\alpha(n) \leqslant e^{-nw(\alpha-\beta)} \sum_{i \in \mathbb{N}} X_i^\beta(n).$$

Choose $\beta = \Upsilon_v + 1$. Thus, as

$$\alpha = C(v)/v + \epsilon = \beta + \epsilon - \Phi(\Upsilon_v)/v$$

and as $\Phi(\Upsilon_v)/v < 0$ when $v > v_{\text{typ}}$ we see that $\beta < \alpha$.

Remark also that

$$(\alpha - \beta)\, v = -\Phi(\beta - 1) + \epsilon v.$$

Thus, if we choose w close enough to v

$$e^{-nw(\alpha-\beta)} \sum_{i \in \mathbb{N}} X_i^\beta(n) = e^{-n\epsilon'} e^{n\Phi(\beta-1)} \sum_{i \in \mathbb{N}} X_i^\beta(n),$$

where $\epsilon' > 0$. Bertoin has shown in ref. 26 (see Theorem 2 therein) that

$$e^{n\Phi(\beta-1)} \sum_{i \in \mathbb{N}} X_i^\beta(n)$$

is a positive martingale. Hence, a.s.

$$e^{-nw(\alpha-\beta)} \sum_{i \in \mathbb{N}} X_i^\beta(n) = o(e^{-n\epsilon'})$$

which concludes the proof.

3.2. Upper Bound for (5)

In this section we consider the case $v < v_{\text{typ}}$. The main difference between the proofs for the upper bounds of (5) and (6) come from the fact that when $v > v_{\text{typ}}$ $\Upsilon_v < 0$ and hence $\Phi(\Upsilon_v) < 0$ whereas the converse is true when $v < v_{\text{typ}}$.

Denote by $\Theta_v^c(t) := \mathbb{N} \setminus \Theta_v(t)$ the complementary in $\mathbb{N}$ of $\Theta_v(t)$. Note that $\forall N \in \mathbb{N}$, for all $w \in]v, v_{\text{typ}}[$ the set $\bigcup_{n \geqslant N} \bigcup_{i \in \Theta_w^c(n)} J_i(n)$ is a cover of $\overline{\mathscr{G}}_v$.

Recall from (7) the notation $I_u(t)$ for the size-biased pick and $\xi_t = -\log |I_u(t)|$ for the associated subordinator. Clearly

$$\mathbf{E}\left[\sum_{i \in \Theta_w^c(t)} X_i^\alpha(t)\right] = \mathbf{E}[\exp(-(\alpha-1)\,\xi_t),\, \xi_t < wt]$$
$$\leqslant e^{awt}\,\mathbf{E}[\exp(-(a+\alpha-1)\,\xi_t)]$$

for all $a > 0$. As $\Upsilon_v > 0$, when $\epsilon = \alpha - C(v)/v < \Phi(\Upsilon_v)/v$ one may choose $a = \Upsilon_v + 1 - \alpha = \Phi(\Upsilon_v)/v - \epsilon > 0$. Hence the right hand term becomes

$$e^{-[\Phi(\Upsilon_v)-(\Upsilon_v+1)\,w+\alpha w]\,t} = e^{-\epsilon v t} e^{(\Upsilon_v+1-\alpha)(w-v)\,t} = e^{-\epsilon' t}$$

for a well chosen $\epsilon' > 0$ when w is close enough to v. Hence the series $\sum_{i \in \Theta_w^c(n)} X_i^\alpha(n)$ is convergent and the upper bound for (5) is proven.

4. LOWER BOUND

To complete the proof of Theorem 1, we wish to construct a subset K of $\mathscr{G}_v$ of Hausdorff dimension large enough. More precisely, we shall obtain a lower bound for $\text{Dim}(K)$ by using the Hölder index of an increasing process indexed by $t \in]0, 1[$ that only grows on points of K, and which can thus be seen as a local time on this set.

We obtain K by mean of a branching process $(G(n))_{n \in \mathbb{N}}$. More precisely $G(n)$ is the union of a collection $H(n)$ of some of the fragments that are present at time δn and that are included in $G(n-1)$, so $(G(n))_{n \in \mathbb{N}}$ is a nested sequence. We will then define $K = \bigcap_{n \in \mathbb{N}} G(n)$. We begin by a careful construction of G. We first define a somewhat "natural" branching process associated to the fragmentation and then show how to modify it to use classical results from the theory of branching processes.

4.1. Construction of the Branching Process

Remark that there is a natural notion of genealogy for interval fragmentations. Namely, the "sons" at time $t+s$ of a given fragment I of $S(s)$

are just the fragments of $S(t+s)$ that are included in I. Our strategy to find points in $\mathscr{G}_v$ will be to look at $S(t)$ at a set of times of the form $\{\delta n\}_{n\in\mathbb{N}}$, and at each step to select the sons of the preceding generation such that the ratio of the sizes father/son lies in an interval above v if the father was too large and under v in the opposite case.

More precisely, take an interval $\kappa \subset]v_{\min}, v_{\max}[$, then for all $t>0$ define

$$\chi_\kappa(t) = \mathrm{Card}\{i: -\log(|J_i(t)|)/t \in \kappa, \{0,1\} \cap \bar{J}_i(t) = \varnothing\}, \tag{9}$$

i.e., the number of intervals with sizes in κ' the image of κ by the map $x \to e^{\epsilon x}$ and which do not touch the boundary of $]0,1[$. The reason why we impose this last condition is that we want to take the intersection of an infinite nested sequence of collections of open intervals. If the closure of each generation is in the interior of the preceding generation and is not empty, then the intersection is not empty. Remark that by homogeneity, for $s>0$ and $j\in\mathbb{N}$, conditionally on $|J_j(s)|>0$, $\chi_\kappa(t)$ has the same law as

$$\mathrm{Card}\left\{\begin{array}{c} J_i(t+s)\subseteq J_j(s), \\ i: -t^{-1}\log\left(\dfrac{|J_i(t+s)|}{|J_j(s)|}\right)\in\kappa, \\ \partial J_j(s)\cap \bar{J}_i(t+s)=\varnothing \end{array}\right\}$$

where ∂I is the boundary of I.

Take $\varepsilon, \delta > 0$ and $H_{\varepsilon,\delta}(0) := \{]0,1[\}$. Define inductively on n the sets $H_{\varepsilon,\delta}(n)$ as the collection of the the interval components of $S(n\delta)$ which fulfill the following three conditions. Firstly, every $I\in H_{\varepsilon,\delta}(n)$ must be included in some $J\in H_{\varepsilon,\delta}(n-1)$. Then there is a relative-size condition: if $I\in H_{\varepsilon,\delta}(n)$ and $J\in H_{\varepsilon,\delta}(n-1)$ are such that $I\subset J$ then

- if $|J| < e^{-v(n-1)\delta}$ then $-\delta^{-1}\log(|I|/|J|)\in[v, v+\varepsilon]$.
- if $|J| \geqslant e^{-v(n-1)\delta}$ then $-\delta^{-1}\log(|I|/|J|)\in[v-\varepsilon, v]$.

In both case we finally impose that $\bar{I}\cap\partial J = \varnothing$.

In some respects $H_{\varepsilon,\delta}$ is much like a multi-type branching process, with each particle corresponding to a segment and thus having some length attached. A "particle" I of the nth generation (i.e., a segment of the collection $H_{\varepsilon,\delta}(n)$) will be called of

- type 1 if $|I| < e^{-vn\delta}$ and in that case its offspring has same distribution as $\chi_{[v,\varepsilon+v]}$
- and of type 2 if $|I| \geqslant e^{-vn\delta}$ and in that case its offspring has same distribution as $\chi_{[v-\varepsilon,v]}$.

The difference being that although here, as in the classical case, the law of the total number of children of a particle I only depends on its type, it happens that the repartition between type 1 and 2 of these children depends on the precise size of I.

However, it can easily be seen by induction that for all $n \in \mathbb{N}$, for any $I_n \in H_{\varepsilon,\delta}(n)$

$$e^{v n\delta}\,|I_n| \in [e^{-\varepsilon\delta}, e^{\varepsilon\delta}]. \tag{10}$$

Thus a.s. for any nested sequence of non-empty intervals $I_n \in H_{\varepsilon,\delta}(n)$ (implicitly, we are conditioning on non-extinction)

$$\lim_{n\to\infty} -\log(|I_n|)/n\delta = v.$$

If there exists $\{x\} = \bigcap I_n$, if we denote by $n_t := \sup\{n \in \mathbb{N} : n\delta < t\}$, then for all t one has the bounds

$$-\log(|I_{n_t}|)/((n_t+1)\,\delta) \leqslant -\log(|I_x(t)|)/t \leqslant -\log(|I_{(n_t+1)}|)/(n_t\delta).$$

Hence, one has that $-\log(|I_x(t)|)/t \to v$ almost surely and $x \in \mathscr{G}_v$.

For all n we define

$$G_{\varepsilon,\delta}(n) := \bigcup_{I \in H_{\varepsilon,\delta}(n)} I,$$

i.e., $H_\cdot(.)$ is the collection of the open interval components of $G_\cdot(.)$. Hence $\bigcap_n G_{\varepsilon,\delta}(n) \subseteq \mathscr{G}_v$. Note that we could not use a monotype branching process here, i.e., at each generation keep the sons such that

$$|I|/|J| \in [e^{-(v+\varepsilon)\,\delta}, e^{-(v-\varepsilon)\,\delta}]$$

because this would lead to points in $\overline{\mathscr{G}}_{v+\varepsilon} \cap \underline{\mathscr{G}}_{v-\varepsilon}$ and not necessarily in $\mathscr{G}_v$.

There are two issues we must take care of now:

- first, we must choose ε and δ such that we catch enough of $\mathscr{G}_v$, and this amounts to control the growth of the branching process
- and second, we would rather work with a true Galton–Watson tree. The branching process $H_{\varepsilon,\delta}$ is the most natural to consider, but in order to use classical results of the branching processes theory we need to cut some branches in order to obtain a true, super-critical, Galton–Watson process. Furthermore, we must do so while keeping its rate of growth close enough to its original value.

4.2. Rate of Growth

In the right-hand side of (8) the interval $[v-\varepsilon, v+\varepsilon]$ is symmetric around v, but it is easy to see that one hardly needs to change the arguments used in ref. 26 to have that a.s.

$$C(v) = \lim_{\varepsilon \to 0} \lim_{t \to \infty} t^{-1} \log \chi_{[v-\varepsilon, v]}(t) \tag{11}$$

and

$$C(v) = \lim_{\varepsilon \to 0} \lim_{t \to \infty} t^{-1} \log \chi_{[v, v+\varepsilon]}(t), \tag{12}$$

where $\chi_{[a,b]}(t)$ is defined by (9).

Hence, clearly if we fix $\epsilon' > 0$ and $\eta > 0$, then we may find $\varepsilon > 0$ and t_0 arbitrarily large such that

$$\forall t > t_0 \colon P(|t^{-1} \log(\chi_{[v-\varepsilon, v]}(t)) - C(v)| > \eta) < \epsilon' \tag{13}$$

and of course the same is true replacing $[v-\varepsilon, v]$ by $[v, v+\varepsilon]$.

For each $t > 0$ consider a variable $\tilde{\chi}(t)$ which law is given by

$$P(\tilde{\chi}(t) = e^{(C(v)-\eta)t}) = 1-\epsilon'$$

and

$$P(\tilde{\chi}(t) = 0) = \epsilon'$$

where

$$\epsilon' = P(|t^{-1} \log(\chi_{[v-\varepsilon, v]}(t)) - C(v)| > \eta) \vee P(|t^{-1} \log(\chi_{[v, v+\varepsilon]}(t)) - C(v)| > \eta).$$

Note that

$$|t^{-1} \log(E[\tilde{\chi}(t)]) - C(v)| \leqslant \eta + t^{-1} |\log(1-\epsilon')|.$$

Fix ϵ and choose ϵ' and η such that $\eta + |\log(1-\epsilon')| < \epsilon$, then choose ε and $t_0 > 1$ by (13).

Plainly $\tilde{\chi}(t)$ is stochastically dominated by $\chi_{[v-\varepsilon, v]}(t)$ and $\chi_{[v, v+\varepsilon]}(t)$. Hence, we can construct a true Galton–Watson tree by thinning $H_{\varepsilon, \delta}$ where $\delta > t_0$. More precisely there exists a procedure for deciding at each node to erase or not some or all of the offspring and such that the resulting tree, denoted by $\mathbb{H}_{v, \epsilon}$ is a Galton–Watson tree, with offspring distribution given by the law of $\tilde{\chi}(\delta)$.

Thus, if we define $m := \mathbf{E}(\tilde{\chi}(\delta))$, the expectation of the number of children of a particle, one has

1.

$$|\delta^{-1} \log m - C(v)| < \epsilon \tag{14}$$

and thus $m > 1$ and $\mathbb{H}_{v,\epsilon}$ is super-critical.

2. for each $n \in \mathbb{N}$ the closure of $\mathbb{G}_{v,\epsilon}(n) := \bigcup_{I \in \mathbb{H}_{v,\epsilon}(n)} I$ is in $\mathbb{G}_{v,\epsilon}(n-1)$

3.

$$\bigcap_{n \in \mathbb{N}} \mathbb{G}_{v,\epsilon}(n) \subseteq \mathscr{G}_v.$$

This last point only makes sense if the tree doesn't die, so in the following we condition systematically on non-extinction.

We now show that $\mathrm{Dim}(\mathscr{G}_v) \geqslant C(v)/v$ for $v \in]v_{\min}, v_{\max}[$ which entails the result for $\overline{\mathscr{G}}_v$ and $\underline{\mathscr{G}}_v$ by inclusion.

4.3. Proof of the Lower Bound

Fix $v \in]v_{\min}, v_{\max}[$ and $\epsilon > 0$. Choose ε and $\delta > t_0$ as shown above and consider the tree $\mathbb{G}_{v,\epsilon}$. Let us recall the signification of the parameters: ϵ controls the precision of the growth rate, ε is the width of the window of acceptable sizes and δ is our time-step. Define

$$Z_{v,\epsilon}(n) = \mathrm{Card}\{\mathbb{H}_{v,\epsilon}(n)\}$$

the size of the nth generation.

In the following to simplify the notations we drop the subscript v, ϵ and we note $Z(n)$, $\mathbb{G}(n)$, or $\mathbb{H}(n)$ for $Z_{v,\epsilon}(n)$, $\mathbb{G}_{v,\epsilon}(n)$, or $\mathbb{H}_{v,\epsilon}(n)$, respectively.

Recall we are conditioning on non-extinction of the branching process $\mathbb{H}(.)$. This conditioning can be made at no cost because in the event that $\mathbb{H}(.)$ becomes extinct, one can restart a new independent tree on any fragment present at the extinction time for instance, and iterate this procedure until one find an infinite tree. Otherwise said, $\bigcap_n G_{\epsilon,\delta}(n)$ is nonempty only with positive probability, but it is however enough to show that its dimension is the correct one only with positive probability, for the dimension must be a constant a.s. due to the independence of fragmentation on different subsets.

It is well known that almost surely

$$m^{-n}Z(n) \to W > 0$$

(more precisely $W > 0$ on the survival set of the tree).

Let σ be a node of our tree (thus it is also a subinterval of $]0, 1[$) and let $|\sigma|$ designate its height in the tree, let $Z^{(\sigma)}(n)$ be the number of its offspring in the tree at the generation $|\sigma|+n$, finally call $W(\sigma)$ the "renormalized weight" of the tree rooted at σ, i.e.,

$$W(\sigma) := \lim_{n\to\infty} m^{-n} \operatorname{Card}\{\sigma' \in \mathbb{H}(|\sigma|+n), \sigma' \subset \sigma\}.$$

Fix an interval $I \subset]0, 1[$ and introduce

$$\mathbb{H}_I(n) = \{\sigma \in \mathbb{H}(n), \sigma \cap I \neq \varnothing\}$$
$$Z_I(n) = \operatorname{Card}(\mathbb{H}_I(n)).$$

Define

$$x \to L_x := \lim_n m^{-n} Z_{]0,\, x[}(n), \qquad x \in]0, 1[.$$

Lemma 2. For each $\epsilon > 0$,

1. there exists a version $\tilde{L}$ of L which is Hölder continuous of order α for any $\alpha < (C(v)-\epsilon)/v$
2. L only grows on the set $\bigcap_n \mathbb{G}_{v,\epsilon}(n)$.

Proof. We show the first point by verifying Kolmogoroff's criterium (see ref. 30, Theorem 2.1, p. 26).

Clearly one has that for all $x < y \in]0, 1[$

$$|L_x - L_y| = \lim_n m^{-n} Z_{]x,\, y[}(n).$$

For any J open subinterval of $]0, 1[$, define

$$\eta(J) := \sup\{n \in \mathbb{N} : e^{-v\delta n} \geqslant |J|\} = [-\log(|J|)/v\delta].$$

This is very close to the largest n for which J can be included in an interval of the collection $\mathbb{H}(n)$, thus it is not difficult to see that at time $\eta(J)$ the number of intervals of $\mathbb{H}(\eta(J))$ which have a non empty intersection with J is bounded. More precisely, according to (10), for each n the size of the intervals in $\mathbb{H}(n)$ have a lower bound given by $e^{-vn\delta-\varepsilon\delta}$, so $|J|\, e^{-(v+\varepsilon)\,\delta}$

is a lower bound for the size of the intervals of $\mathbb{H}(\eta(J))$, and thus $Z_J(\eta(J)) \leqslant e^{(v+\varepsilon)\delta}$.

Thus, for all x, y such that $x < y$, one has by definition of L_x and using (14) that

$$\begin{aligned} |L_y - L_x| &\leqslant m^{-\eta(]x,\,y[)} \sum_{\sigma \in \mathbb{H}_{]x,\,y[}(\eta(]x,\,y[))} W(\sigma) \\ &\leqslant e^{\log m(\frac{1}{v\delta}\log(y-x)+1)} \sum_{\sigma \in \mathbb{H}_{]x,\,y[}(\eta(]x,\,y[))} W(\sigma) \\ &\leqslant m\,|y-x|^{(C(v)-\epsilon)/v} \sum_{\sigma \in \mathbb{H}_{]x,\,y[}(\eta(]x,\,y[))} W(\sigma). \end{aligned}$$

Remark that for all $\gamma > 1$ and all $J \subset]0, 1[$ one has

$$\mathbf{E}\left[\left(\sum_{\sigma \in \mathbb{H}_J(\eta(J))} W(\sigma)\right)^{\gamma}\right] < \mathbf{E}[(W_1 + W_2 + \cdots + W_{\lceil e^{(v+\varepsilon)\delta}\rceil+1})^{\gamma}] < \infty$$

where the W_i are independent copies of W and the finiteness comes from the existence of finite moments of all order for W (this follows from, e.g., Biggins and Bingham[31]).

We conclude that for each $\gamma > 1$ there exists a $k > 0$ such that

$$\mathbf{E}(|L_y - L_x|^{\gamma}) \leqslant k\,|y-x|^{\gamma(C(v)-\epsilon)/v}$$

which proves our first assertion.

The second part of the lemma is straightforward. The increasing function L_x only grows on the points of $]0, 1[$ that correspond to the frontier of the tree, i.e., on $\bigcap_n \mathbb{G}(n)$. More precisely, for any interval $]a, b[\subset (\bigcap_n \mathbb{G}(n))^c$ one clearly has that $L_a = L_b$ by definition. On the other hand $L_{0+} = 0$ and $L_{1-} = W > 0$ so L_x can be thought of as a local time on $\bigcap_n \mathbb{G}(n)$. ∎

We conclude with the proof of Theorem 1. For a cover of $\bigcap_n G(n)$ of the type $\bigcup_{i=1}^{N}]l_i, r_i[$ (where the $]l_i, r_i[$ are disjoints open intervals) one has a.s.

$$\sum_{i=1}^{N} |\tilde{L}_{r_i} - \tilde{L}_{l_i}| = W.$$

Thus for all such cover with $\max_i(r_i - l_i)$ small enough

$$W \leqslant k \sum_{i=1}^{N} (r_i - l_i)^{(C(v)-\epsilon)/v}$$

and hence a.s.

$$\mathrm{Dim}(\mathscr{G}_v) \geqslant \mathrm{Dim}\left(\bigcap_n (G(n))\right) \geqslant (C(v)-\epsilon)/v.$$

To conclude simply let $\epsilon \to 0$.

Alternatively one could use the same approach as Orey and Taylor in ref. 11. They apply a lemma for the lower bound which is specific to self-similar sets, but as the arguments are very similar to those above we do not include this version of the proof. There is also a way of doing the proof by using some fine results of Liu[20] on the local behavior of the branching measure that does not rely on the Kolmogorov criterium.

We conclude with the three following remarks.

Remark 1. Although formally $\underline{p} \geqslant -1$, in the cases analogue to the introductory example (i.e., when the fragmentation is slow enough and Φ can be analytically extended beyond -1) then it can be shown that the theorem holds for some v larger than $\Phi'(\underline{p}+)$, more precisely for $v \in]v_{\min}, \Phi'(p^*)[$ where Φ is extended and

$$p^* = \inf\{p: 1 + p^* - \Phi(p^*)/\Phi'(p^*) > 0\}.$$

When it is finite, $\Phi'(p^*)$ is the maximum fragmentation speed.

Remark 2. In Theorem 1, relations (5) and (6) hold almost surely simultaneously for all v. Indeed, call Ω_0 a set of probability 1 on which (4), (5), and (6) are true for each v rational. Define the set of events

$$A := \{\exists v \in]v_{\min}, v_{\max}[\text{ s.t. (5) or (6) is not true}\}.$$

As for each $v \leqslant w$ one has $\overline{\mathscr{G}}_v \subset \overline{\mathscr{G}}_w$ and $\underline{\mathscr{G}}_w \subset \underline{\mathscr{G}}_v$, it is clear that $A \subset \Omega_0^c$ the complementary of Ω_0, and hence $P(A) = 0$. The same arguments show that almost surely for all $v \in]v_{\min}, v_{\mathrm{typ}}[$ one has

$$\mathrm{Dim}(\mathscr{G}_v) \leqslant C(v)/v,$$

and almost surely for all $v \in]v_{\mathrm{typ}}, v_{\max}[$ one has

$$\mathrm{Dim}(\mathscr{G}_v) \geqslant C(v)/v.$$

Although it seems doubtful that there exists an exceptional set of v's for which (4) does not holds, the techniques employed in the present work does not allow one to conclude on that matter.

Remark 3. Define

$$\bar{\mathscr{H}}_v := \left\{ x \in]0, 1[: \liminf_{t \to \infty} -t^{-1} \log(|I_x(t)|) \leqslant v \right\},$$

$$\underline{\mathscr{H}}_v := \left\{ x \in]0, 1[: \limsup_{t \to \infty} -t^{-1} \log(|I_x(t)|) \geqslant v \right\}.$$

Clearly $\bar{\mathscr{G}}_v \subset \bar{\mathscr{H}}_v$ and $\underline{\mathscr{G}}_v \subset \underline{\mathscr{H}}_v$, thus

$$\mathrm{Dim}(\bar{\mathscr{H}}_v) \geqslant \mathrm{Dim}(\bar{\mathscr{G}}_v)$$

and

$$\mathrm{Dim}(\underline{\mathscr{H}}_v) \geqslant \mathrm{Dim}(\underline{\mathscr{G}}_v).$$

Furthermore, it is easy to see by inspection of the proof in Section 3 that the the same upper bound holds for $\bar{\mathscr{H}}_v$ (resp. $\underline{\mathscr{H}}_v$) and $\bar{\mathscr{G}}_v$ (resp. $\underline{\mathscr{G}}_v$). Thus

$$\mathrm{Dim}(\bar{\mathscr{H}}_v) = \mathrm{Dim}(\bar{\mathscr{G}}_v)$$

and

$$\mathrm{Dim}(\underline{\mathscr{H}}_v) = \mathrm{Dim}(\underline{\mathscr{G}}_v).$$

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